

Existence of positive solution for a system of elliptic equations via bifurcation theory

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Abstract

In this paper we study the existence of solution for the following class of system of elliptic equations

$$\begin{cases} -\Delta u = (a - \int_{\Omega} K(x, y)f(u, v)dy)u + bv, & \text{in } \Omega \\ -\Delta v = (d - \int_{\Omega} \Gamma(x, y)g(u, v)dy)v + cu, & \text{in } \Omega \\ u = v = 0, & \text{on } \partial\Omega \end{cases} \quad (P)$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geq 1$, and $K, \Gamma : \Omega \times \Omega \rightarrow \mathbb{R}$ is a nonnegative function checking some hypotheses and $a, b, c, d \in \mathbb{R}$. The functions f and g satisfy some conditions which permit to use Bifurcation Theory to prove the existence of solution for (P) .

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1 Introduction and main result

The main goal of this paper is to study the existence of positive solution for the following class of nonlocal problems

$$\begin{cases} -\Delta u = (a - \int_{\Omega} K(x, y)f(u, v)dy)u + bv, & \text{in } \Omega \\ -\Delta v = (d - \int_{\Omega} \Gamma(x, y)g(u, v)dy)v + cu, & \text{in } \Omega \\ u = v = 0, & \text{on } \partial\Omega \end{cases} \quad (P)$$

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where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geq 1$ and $K, \Gamma : \Omega \times \Omega \rightarrow \mathbb{R}$ are nonnegative functions checking some hypotheses and $a, b, c, d \in \mathbb{R}$. The functions f and g satisfy some technical conditions which will be mentioned later on.

The study of the problem (P) comes from the problem to model the behavior of a species inhabiting in a smooth bounded domain $\Omega \subset \mathbb{R}^N$, whose the classical logistic equation is given by

$$\begin{cases} -\Delta u = u(\lambda - b(x)u^p), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \quad (1)$$

where $u(x)$ is the population density at location $x \in \Omega$, $\lambda \in \mathbb{R}$ is the growth rate of the species, and b is a positive function denoting the carrying capacity, that is, $b(x)$ describes the limiting effect of crowding of the population.

Since (1) is a local problem, the crowding effect of the population u at x only depends on the value of the population in the same point x . In [5], for more realistic situations, Chipot has considered that the crowding effect depends also on the value of the population around of x , that is, the crowding effect depends on the value of integral involving the function u in the ball $B_r(x)$ centered at x of radius $r > 0$. To be more precisely, in [5] the following nonlocal problem has been studied

$$\begin{cases} -\Delta u = \left(\lambda - \int_{\Omega \cap B_r(x)} b(y)u^p(y)dy \right) u, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \quad (2)$$

where b is a nonnegative and nontrivial continuous function. After [5], a special attention has been given for the problem

$$\begin{cases} -\Delta u = \left(\lambda - \int_{\Omega} K(x, y)u^p(y)dy \right) u, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \quad (3)$$

by supposing different conditions on K , see for example, Allegretto and Nistri [1], Alves, Delgado, Souto and Suárez [2], Chen and Shi [4], Corrêa, Delgado and Suárez [6], Coville [9], Leman, Méléard and Mirrahimi [13], and Sun, Shi and Wang [16] and their references.

In [2], Alves, Delgado, Souto and Suárez have considered the existence and nonexistence of solution for Problem (3). In the paper, the authors have introduced a class \mathcal{K} which is formed by functions $K : \Omega \times \Omega \rightarrow \mathbb{R}$ such that:

(i) $K \in L^\infty(\Omega \times \Omega)$ and $K(x, y) \geq 0$ for all $x, y \in \Omega$.

(ii) If w is measurable and $\int_{\Omega \times \Omega} K(x, y)|w(y)|^p|w(x)|^2 dx dy = 0$, then $w = 0$ a.e. in Ω .

Using Bifurcation Theory and by supposing that K belongs to class \mathcal{K} , the following result has been proved

Theorem 1.1. *The problem (3) has a positive solution if, and only if, $\lambda > \lambda_1$, where λ_1 is the first eigenvalue of problem*

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Motivated by [2], at least from a mathematical point of view, it seems to be interesting to ask if

$$\begin{cases} -\Delta u = (\lambda f(x) - \int_{\mathbb{R}^N} K(x, y) |u(y)|^\gamma dy) u, & \text{in } \mathbb{R}^N \\ \lim_{|x| \rightarrow +\infty} u(x) = 0, \quad u > 0 & \text{in } \mathbb{R}^N \end{cases} \quad (Q)$$

version in \mathbb{R}^N for (3), has a solution. This question was answered by Alves, de Lima and Souto in [3]. In this paper, the authors study the existence of positive solution for (Q).

Using Bifurcation Theory and inspired by the results due to Edelson and Rumbos [11, 12], Alves, de Lima and Souto, have shown that under some conditions on K and f , problem (Q) has a positive solution if, and only if, $\lambda > \lambda_1$, where λ_1 is the first eigenvalue of the linear problem

$$\begin{cases} -\Delta u = \lambda f(x)u, & \text{in } \mathbb{R}^N \\ \lim_{|x| \rightarrow +\infty} u(x) = 0. \end{cases} \quad (AQ)$$

This result and your proof can be found in [3], where the reader can find the assumptions on K and f .

Motivated by [2], comes a new challenge: model the behavior of two species inhabiting in a smooth bounded domain $\Omega \subset \mathbb{R}^N$, analogous modeling done in the case of a species in [2]. Inspired in the articles due to Corrêa and Souto [7, 8] and Souto [15], we propose the following system to model the problem

$$\begin{cases} -\Delta u = (a - \int_{\Omega} K(x, y) f(u, v) dy) u + bv, & \text{in } \Omega \\ -\Delta v = (d - \int_{\Omega} \Gamma(x, y) g(u, v) dy) v + cu, & \text{in } \Omega \\ u = v = 0, & \text{on } \partial\Omega. \end{cases} \quad (P)$$

It is very interesting to note that in a situation where $a, b, c, d > 0$, we are in a cooperative system, i.e., the two species involved mutually cooperate to their growth. If $b \cdot c < 0$, we say that we are in a structure involving predator and prey. In which case $b, c < 0$, there is a competition between the two species.

In the present article, well as in [2], the class \mathcal{K} is formed by functions $K : \Omega \times \Omega \rightarrow \mathbb{R}$ such that

i) $K \in L^\infty(\Omega \times \Omega)$ and $K(x, y) \geq 0$ for all $x, y \in \Omega$.

ii) If w is a measurable function and $\int_{\Omega \times \Omega} K(x, y) |w(y)|^\gamma w(x)^2 dx dy = 0$, then $w = 0$ a.e. in Ω .

The functions $K : \Omega \times \Omega \rightarrow \mathbb{R}$ and $\Gamma : \Omega \times \Omega \rightarrow \mathbb{R}$ that we are considering belong to class \mathcal{K} .

Related to functions f and g , we assume that

(f_0) $f, g : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}^+$ are continuous functions.

(f_1) There exists $\epsilon > 0$ such that $f(t, s) \geq \epsilon|t|^\gamma$ and $g(t, s) \geq \epsilon|s|^\gamma$, for all $t, s \in [0, \infty)$ and $\gamma > 0$.

(f_2) $f(pt, ps) = p^\gamma f(t, s)$ and $g(pt, ps) = p^\gamma g(t, s)$, for all $t, s \in [0, \infty)$ e $p > 0$, where $\gamma > 0$.

(f_3) There exists $c > 0$ such that $f(t, s), g(t, s) \leq c$, always that $|(t, s)| \leq 1$.

The functions $f(t, s) = |t|^\gamma + |s|^{\gamma-\mu}|t|^\mu$ and $g(t, s) = c_1|t|^\gamma + c_2|s|^\gamma$ are examples that verifies (f_0)–(f_3).

The constants $a, b, c, d \in \mathbb{R}$, that appear in the system (P), forming the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, which often appears in this article.

Our main results are the following:

Theorem 1.2. *For a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d > 0$ and $\lambda > 0$ its largest eigenvalue. We have that, the problem*

$$\begin{cases} -\Delta u = (a - \int_{\Omega} K(x, y)f(u, v)dy)u + bv, & \text{in } \Omega \\ -\Delta v = (d - \int_{\Omega} \Gamma(x, y)g(u, v)dy)v + cu, & \text{in } \Omega \\ u, v > 0, & \text{in } \Omega \\ u = v = 0, & \text{on } \partial\Omega. \end{cases} \quad (P_1)$$

has solution if, and only if, $\lambda > \lambda_1$, where λ_1 is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$.

In which case $f = g$ and $K = \Gamma$, we have:

Theorem 1.3. *Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ a matrix such that: there is a positive and largest eigenvalue of A that is the unique positive eigenvalue λ with an eigenvector $z > 0$ and $\dim N(\lambda I - A) = 1$. Then, the problem*

$$\begin{cases} -\Delta u = (a - \int_{\Omega} K(x, y)f(u, v)dy)u + bv, & \text{in } \Omega \\ -\Delta v = (d - \int_{\Omega} K(x, y)f(u, v)dy)v + cu, & \text{in } \Omega \\ u, v > 0, & \text{in } \Omega \\ u = v = 0, & \text{on } \partial\Omega. \end{cases} \quad (P_2)$$

has solution for all $\lambda > \lambda_1$, where λ_1 is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$.

Notations

- $\sigma(A)$ denotes the set of real eigenvalues of the matrix A .
- $\sigma(-\Delta)$ denotes the set of eigenvalues of the operator $(-\Delta, H_0^1(\Omega))$, which has its notations and properties already well known.
- The terms of the form $U = (u, v)$, whenever it is convenient, will be written in column matrix form $U = \begin{pmatrix} u \\ v \end{pmatrix}$. Moreover, $-\Delta U = (-\Delta u, -\Delta v)$ or $-\Delta U = \begin{pmatrix} -\Delta u \\ -\Delta v \end{pmatrix}$.
- E denotes the Banach space $C(\overline{\Omega}) \times C(\overline{\Omega})$, with norm given by

$$\|U\| = \|u\|_{C(\overline{\Omega})} + \|v\|_{C(\overline{\Omega})}$$

where $U \in E$, that will always be denoted by $U = (u, v)$ or, in the column matrix form, $U = \begin{pmatrix} u \\ v \end{pmatrix}$, for $u, v \in C(\overline{\Omega})$.

- E_1 denotes the Banach space $C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$, with norm given by

$$\|U\|_1 = \|u\|_{C^1(\overline{\Omega})} + \|v\|_{C^1(\overline{\Omega})}$$

where $U \in E_1$, that will always be denoted by $U = (u, v)$ or, in the column matrix form, $U = \begin{pmatrix} u \\ v \end{pmatrix}$, for $u, v \in C^1(\overline{\Omega})$.

- $z = (\alpha, \beta) > 0$ or $z = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} > 0$, denotes that $\alpha, \beta > 0$.

2 The nonlocal terms and the matricial formulation

Suposing that $K, \Gamma \in \mathcal{K}$, as in all text, f and g check $(f_0) - (f_3)$, is well defined $\phi, \psi : L^\infty(\Omega) \times L^\infty(\Omega) \rightarrow L^\infty(\Omega)$ given by

$$\phi_{(u,v)}(x) = \int_{\Omega} K(x, y) f(|u(y)|, |v(y)|) dy$$

and

$$\psi_{(u,v)}(x) = \int_{\Omega} \Gamma(x, y) g(|u(y)|, |v(y)|) dy.$$

Moreover, using the hypothesis of K, Γ, f and g , we have the properties:

$$(\phi_1) \quad t^\gamma \phi_{(u,v)} = \phi_{(tu, tv)} \quad \text{and} \quad t^\gamma \psi_{(u,v)} = \psi_{(tu, tv)}, \quad \text{for all } u, v \in L^\infty(\Omega), \quad t > 0 \text{ and } \gamma > 0.$$

(ϕ_2) $\|\phi_{(u,v)}\|_\infty \leq \|K\|_\infty |\Omega| \|f(|u|, |v|)\|_\infty$ and $\|\psi_{(u,v)}\|_\infty \leq \|K\|_\infty |\Omega| \|g(|u|, |v|)\|_\infty$, for all $u, v \in L^\infty(\Omega)$.

With these notations, we fix:

$$\Phi_U(x) := \begin{pmatrix} u\phi_{(u,v)}(x) \\ v\psi_{(u,v)}(x) \end{pmatrix}, \quad \text{where } U = (u, v) \in L^\infty(\Omega) \times L^\infty(\Omega).$$

Using the established and fixed above notations, the problem (P_1) can be written in the form:

$$\begin{cases} -\Delta U + \Phi_U(x) = AU, & \text{in } \Omega \\ U > 0, & \text{in } \Omega \\ U = 0, & \text{on } \partial\Omega \end{cases} \quad (P_3)$$

or equivalently,

$$\begin{cases} -\Delta u + \phi_{(u,v)}u = au + bv, & \text{in } \Omega \\ -\Delta v + \psi_{(u,v)}u = cu + dv, & \text{in } \Omega \\ u, v > 0, & \text{in } \Omega \\ u = v = 0, & \text{on } \partial\Omega. \end{cases} \quad (P_4)$$

Here, we recall that $U = (u, v)$ satisfies the above problem in the weak sense, if $u, v \in H_0^1(\Omega)$ and

$$\int_\Omega \nabla u \nabla \varphi dx + \int_\Omega \phi_{(u,v)}(x) u \varphi dx = \int_\Omega (au + bv) \varphi dx \quad (4)$$

$$\int_\Omega \nabla v \nabla \eta dx + \int_\Omega \psi_{(u,v)}(x) v \eta dx = \int_\Omega (cu + dv) \eta dx \quad (5)$$

for all $\eta, \varphi \in H_0^1(\Omega)$.

In which case $f = g$ and $K = \Gamma$, we have $\phi_{(u,v)} = \psi_{(u,v)}$ and, consequently, $\Phi_U(x) = \phi(x)U$, where $\phi(x) = \phi_{(u,v)}(x)$. Thus, the problem (P_2) can be written in the form

$$\begin{cases} -\Delta U + \phi(x)U = AU, & \text{in } \Omega \\ U > 0, & \text{in } \Omega \\ U = 0, & \text{on } \partial\Omega. \end{cases} \quad (P_6)$$

3 Technical results

From characteristic of our problem, it is necessary to make a technical study of matrices that include the our study. This section is developed to present this study, which is essential in all text.

Lemma 3.1. Suppose there is a solution $U = \begin{pmatrix} u \\ v \end{pmatrix}$ nontrivial for the homogeneous system

$$\begin{cases} -\Delta U &= AU, & \text{in } \Omega \\ U &= 0, & \text{on } \partial\Omega. \end{cases} \quad (Q_1)$$

Then A has a real eigenvalue which is also an eigenvalue $(-\Delta, H_0^1(\Omega))$. Furthermore:

- i) if $\lambda_j \in \sigma(-\Delta) \cap \sigma(A)$, for ϕ_j eigenfunction of $(-\Delta, H_0^1(\Omega))$ associated with the eigenvalue λ_j , we have that $z = \begin{pmatrix} \int_{\Omega} u \phi_j dx \\ \int_{\Omega} v \phi_j dx \end{pmatrix}$ is eigenvector of A associated with the eigenvalue λ_j .
- ii) if $\sigma(-\Delta) \cap \sigma(A) = \{\lambda_j\}$ and $\dim N(A - \lambda_j I) = 1$, then every solution of (Q_1) is of the form $U = \phi_j z$, where z is an eigenvector of A associated with λ_j . Moreover, the subspace $N_A = \{U \in E; U \text{ is a solution of the problem } (Q_1)\}$ has the same dimension of the eigenspace associated with λ_j as eigenvalue of $(-\Delta, H_0^1(\Omega))$.
- iii) if $\sigma(-\Delta) \cap \sigma(A) = \{\lambda_j, \lambda_m\}$, $m \neq j$, then every solution of (Q_1) is of the form $U = \phi_j z + \phi_m w$, where z is an eigenvector of A associated with λ_j and w is an eigenvector of A associated with λ_m . In this case, $\dim N_A$ is the sum of the dimension of the associated eigenspace with λ_j as eigenvalue of $(-\Delta, H_0^1(\Omega))$ and the dimension of the associated eigenspace with λ_m as eigenvalue of $(-\Delta, H_0^1(\Omega))$.

Lemma 3.2. Suppose that, there is a solution $U = \begin{pmatrix} u \\ v \end{pmatrix}$ nonnegative and nonzero for the homogeneous system

$$\begin{cases} -\Delta U &= AU, & \text{in } \Omega \\ U &= 0, & \text{on } \partial\Omega. \end{cases} \quad (Q_1)$$

Then, A has λ_1 as one of the eigenvalues, which has an eigenvector associated with positive coordinates.

Corollary 3.1. If $\sigma(A) = \{\mu, \lambda\}$, $\mu < \lambda$, $\lambda > 0$ and $z > 0$ is an eigenvector of A associated to eigenvalue λ . Then, if (Q_1) has U as nonnegative and nonzero solution, we have $\lambda = \lambda_1$ and $U = \phi_1 w$, where w is multiple of z . Moreover, we have that $U > 0$ and $\frac{\partial u}{\partial \eta}, \frac{\partial v}{\partial \eta} < 0$ on $\partial\Omega$.

For the reader's convenience, we present a sketch of the proofs of these above results in the appendix.

3.1 The parameter t in the homogeneous problem

Assuming that the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has a positive λ eigenvalue associated to a positive $z = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ eigenvector, that is, $Az = \lambda z$ with $\alpha, \beta > 0$.

Our interesting is to give hypotheses about $t > 0$ for that the system

$$\begin{cases} -\Delta U &= tAU, & \text{in } \Omega \\ U &= 0, & \text{on } \partial\Omega. \end{cases}$$

has a space of one-dimensional solutions and a solution $U > 0$ in Ω .

Assuming the existence of a positive eigenvalue λ , consider $t = t_1 = \frac{\lambda_1}{\lambda}$. And so, $t_1 Az = t_1 \lambda z = \lambda_1 z$. Clearly, $U = \begin{pmatrix} \alpha \phi_1 \\ \beta \phi_1 \end{pmatrix}$ is positive and satisfies $-\Delta U = t_1 AU$. Therefore, the space of solutions to the problem for $t = t_1$, $N_1 = N_{(t_1 A)}$ has positive dimension.

In order to have a space of solutions with dimension one, we consider the situation:

- if $\sigma(A) = \{\lambda, \mu\}$, with $\lambda_1 \mu \neq \lambda_j \lambda$, for all $j > 1$. In this case, $\sigma(t_1 A) = \{t_1 \mu, \lambda_1\}$. As $t_1 \mu \neq \lambda_j$, for all $j > 1$, follow that $\dim N_1 = 1$.

It is easy to see that if $\lambda > \mu$, the above condition is always satisfied: $t_1 \mu < t_1 \lambda = \lambda_1 < \lambda_j$, for all $j > 1$.

The situation here descript is utilized in the Theorems 1.2 and 1.3.

On the other hand, since we will make use of the global bifurcation theorem, note that if A has two positive eigenvalues λ and μ , and each is associated positive eigenvectors z and w , respectively, then $\dim N_{(tA)} = 1$, for $t = t_1$ and $t = s_1$, where $t_1 = \lambda_1/\lambda$ and $s_1 = \lambda_1/\mu$. Moreover, $0 < z\phi_1 \in N_{(t_1 A)}$ and $0 < w\phi_1 \in N_{(s_1 A)}$. Thus, a bifurcation can starts at $t = t_1$ and may finish in $t = s_1$. We must avoid this situation.

Therefore, the hypothesis on A is that this matrix has at least one positive eigenvalue λ with $\dim N(A - \lambda I) = 1$, and it is associated with a positive eigenvector z . Moreover, if A has another positive eigenvalue μ , must be associated with an eigenvector $w = \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$ with $\alpha_2 \beta_2 < 0$.

By Lemma 5.2, a bifurcation with positive solutions should start in $t = t_1$.

Remark 3.1. *Recalling linear algebra of the matrices 2×2*

For a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d > 0$, it is possible to prove that $\sigma(A) = \{\mu, \lambda\}$ and $\lambda > \mu$ with $\lambda > 0$. It is well known that there exist $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ eigenvector of A associated with the eigenvalue λ with $\alpha, \beta > 0$ and $\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}$ eigenvector of A associated with the eigenvalue μ with $\alpha_1 \beta_1 < 0$.

4 Comments on the solutions operators

We intend to prove the existence of positive solution for (P_1) and (P_2) by using the classical bifurcation result due to Rabinowitz, see [14]. To this end, we recall that there exists $c_\infty = c_\infty(\Omega) > 0$ such that: for each $h \in L^\infty(\Omega)$, there is only $\omega \in C^1(\overline{\Omega})$ satisfying:

$$\begin{cases} -\Delta \omega = h(x), & \text{in } \Omega \\ \omega = 0, & \text{on } \partial\Omega \end{cases}$$

and

$$\|\omega\|_{C^1(\overline{\Omega})} \leq c_\infty \|h\|_\infty.$$

We use the property up freely in the construction of elementary properties of operators who build below.

Considering the solution operator $S : E \rightarrow E_1$, given by

$$S(u, v) = (u_1, v_1) \Leftrightarrow \begin{cases} -\Delta u_1 = au + bv, & \text{in } \Omega \\ -\Delta v_1 = cu + dv, & \text{in } \Omega \\ u_1 = v_1 = 0, & \text{on } \partial\Omega \end{cases}$$

or, equivalently, in the matricial form

$$S(U) = U_1 \Leftrightarrow \begin{cases} -\Delta U_1 = AU, & \text{in } \Omega \\ U_1 = 0, & \text{on } \partial\Omega \end{cases}$$

where $U = \begin{pmatrix} u \\ v \end{pmatrix}$ and $U_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$. We have that, S is well defined, is linear and verifies

$$\|S(U)\|_1 \leq C\|U\|, \text{ for all } U \in E.$$

Moreover, using the Schauder embedding $S : E \rightarrow E$ is a compact operator.

On the other hand, setting the nonlinear operator $G : E \rightarrow E_1$ given by

$$G(u, v) = (u_1, v_1) \Leftrightarrow \begin{cases} -\Delta u_1 + \phi_{(u,v)}(x)u = 0, & \text{in } \Omega \\ -\Delta v_1 + \psi_{(u,v)}(x)v = 0, & \text{in } \Omega \\ u_1 = v_1 = 0, & \text{on } \partial\Omega \end{cases}$$

or, equivalently, in matricial form

$$G(U) = U_1 \Leftrightarrow \begin{cases} -\Delta U_1 + \Phi_U(x) = 0, & \text{in } \Omega \\ U_1 = 0, & \text{on } \partial\Omega \end{cases}$$

where $\Phi_U(x) = \begin{pmatrix} u\phi_{(u,v)}(x) \\ v\psi_{(u,v)}(x) \end{pmatrix}$ and $U = (u, v)$. We have, clearly, that G is well defined, it is continuous and checks

$$\|G(U)\|_1 \leq C(\|\phi_{(u,v)}\|_\infty + \|\psi_{(u,v)}\|_\infty)\|U\|, \text{ for all } U \in E.$$

Using again the Schauder embedding, we have that $G : E \rightarrow E$ is compact. Moreover, from $(f_2) - (f_3)$ and (ϕ_2) , is possible to verify that

$$G(U) = o(\|U\|).$$

5 Proof of Theorem 1.2

In order to prove Theorem 1.2 via bifurcation theory, it is necessary to introduce a parameter $t > 0$ in the problem (P_1) and prove the lemma below:

Lemma 5.1. *For a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d > 0$ and $\lambda > 0$ its largest eigenvalue. We have that, the problem*

$$\begin{cases} -\Delta U + \Phi_U(x) = tAU, & \text{in } \Omega \\ U > 0, & \text{in } \Omega \\ U = 0, & \text{on } \partial\Omega \end{cases} \quad (P_7)$$

has solution if, and only if, $t > t_1$, where $t_1 = \frac{\lambda_1}{\lambda}$ and λ_1 is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$.

To prove the lemma above, it is necessary to note that: using the definitions of S and G , it is easy to check that $(t, U) \in \mathbb{R} \times E$ solves (P_7) if, and only if,

$$U = F(t, U) = tS(U) + G(U).$$

In the sequel, we will apply the following result due to Rabinowitz [14], to prove the Lemma 5.1.

Theorem 5.1. (Global bifurcation) *Let E be a Banach space. Suppose that S is a compact linear operator and $t^{-1} \in \sigma(S)$ has odd algebraic multiplicity. If G is a compact operator and*

$$\lim_{\|u\| \rightarrow 0} \frac{G(u)}{\|u\|} = 0,$$

then the set

$$\Sigma = \overline{\{(t, u) \in \mathbb{R} \times E : u = tS(u) + G(u), u \neq 0\}}$$

has a closed connected component $\mathcal{C} = \mathcal{C}_t$ such that $(t, 0) \in \mathcal{C}$ and

(i) \mathcal{C} is unbounded in $\mathbb{R} \times E$, or

(ii) there exists $\hat{t} \neq t$, such that $(\hat{t}, 0) \in \mathcal{C}$ and $\hat{t}^{-1} \in \sigma(S)$.

By study done in Subsection 3.1, an eigenfunction U_1 associated with eigenvalue $t_1 = \lambda_1/\lambda$ of the linear problem can be chosen positive. In addition, t_1^{-1} is an eigenvalue of multiplicity 1 for S . From global bifurcation theorem, there exists a closed connected component $\mathcal{C} = \mathcal{C}_{t_1}$ of solutions for (P_7) , which satisfies (i) or (ii). We claim that (ii) does not occur. In order to show this claim, we need the two lemmas below

Lemma 5.2. *There exists $\delta > 0$ such that, if $(t, U) \in \mathcal{C}$ with $|t - t_1| + \|U\| < \delta$ and $U \neq 0$, then U has defined signal, that is,*

$$U(x) > 0, \quad \forall x \in \Omega \quad \text{or} \quad U(x) < 0, \quad \forall x \in \Omega.$$

Proof. It is enough to prove that for any two sequences $(U_n) \subset E$ and $t_n \rightarrow t_1$ with

$$U_n \neq 0, \quad \|U_n\| \rightarrow 0 \quad \text{and} \quad U_n = F(t_n, U_n) = t_n S(U_n) + G(U_n),$$

U_n has defined signal for n large enough.

Setting $W_n = U_n/\|U_n\|$, we have that

$$W_n = t_n S(W_n) + \frac{G(U_n)}{\|U_n\|} = t_n S(W_n) + o_n(1).$$

From compactness of the operator S , we can assume that $(S(W_n))$ is convergent. Then, $W_n \rightarrow W$ in E for some $W \in E$ with $\|W\| = 1$. Consequently,

$$\begin{cases} -\Delta W &= t_1 A W, & \text{in } \Omega \\ W &= 0, & \text{on } \partial\Omega. \end{cases}$$

Once that $W \neq 0$, we have, by Lemmas 3.1 and 3.2, that

$$W(x) > 0 \text{ or } W(x) < 0, \text{ for all } x \in \Omega.$$

Therefore, without loss of generality, $W > 0$ in Ω , and consequently $W_n > 0$ in Ω for n large enough. Once U_n and W_n has the same signal, we have that U_n is also positive, this completes to proof. ■

It is easy to check that if $(t, U) \in \Sigma$, the pair $(t, -U) \in \Sigma$. From maximum principle arguments used in [2] and positivity of a, b, c and d , we can decompose \mathcal{C} in $\mathcal{C}^+ \cup \mathcal{C}^-$, where

$$\mathcal{C}^+ = \{(t, U) \in \mathcal{C}; U > 0\} \cup \{(t_1, 0)\}$$

and

$$\mathcal{C}^- = \{(t, U) \in \mathcal{C}; U < 0\} \cup \{(t_1, 0)\}.$$

Observed that, $\mathcal{C}^- = \{(t, U) \in \mathcal{C}; (t, -U) \in \mathcal{C}^+\}$, $\mathcal{C}^+ \cap \mathcal{C}^- = \{(t_1, 0)\}$ and \mathcal{C}^+ is unbounded if, and only if, \mathcal{C}^- is unbounded.

Now, we are able to prove that (ii) does not hold.

Lemma 5.3. *\mathcal{C}^+ is unbounded.*

Proof. Suppose by contradiction that \mathcal{C}^+ is bounded. Then, \mathcal{C} is also bounded. From global bifurcation theorem, there exists $(\hat{t}, 0) \in \mathcal{C}$, where $\hat{t} \neq t_1$ e $\hat{t}^{-1} \in \sigma(S)$.

Hence, without loss of generality, there exist $(t_n, U_n) \in \mathcal{C}^+$ with $t_n \rightarrow \hat{t}$ such that

$$U_n \neq 0, \quad \|U_n\| \rightarrow 0 \text{ and } U_n = F(t_n, U_n).$$

Setting $W_n = U_n / \|U_n\|$, similar to what was done in the previous lemma, there exists $W \in E$ with $W_n \rightarrow W$ in E , where $W \neq 0$, $W \geq 0$ and satisfies

$$\begin{cases} -\Delta W &= (\hat{t}A)W, & \text{in } \Omega \\ W &= 0, & \text{in } \partial\Omega. \end{cases}$$

From Corollary 3.1, $\hat{t}\lambda = \lambda_1$ and, consequently, $\hat{t} = t_1$, which is impossible. This proves the lemma. ■

From previous lemma, the connected component \mathcal{C}^+ is unbounded. Now, our goal is to show that this component intersects any hyperplane $\{t\} \times E$, for $t > t_1$. To see this, we need of the following a priori estimate

Lemma 5.4. (*A priori estimate*) *For any $\Lambda > 0$, there exists $R > 0$ such that, if $(t, U) \in \mathcal{C}^+$ and $t \in [0, \Lambda]$, then $\|U\| \leq R$.*

Proof. Setting by $\|\cdot\|_H$, the norm in $H = H_0^1(\Omega) \times H_0^1(\Omega)$, given by

$$\|U\|_H = \|u\|_{H_0^1(\Omega)} + \|v\|_{H_0^1(\Omega)}$$

where $U = \begin{pmatrix} u \\ v \end{pmatrix}$ with $u, v \in H_0^1(\Omega)$.

We start showing an a priori estimate on the H space:

Claim 5.1. *Given $\Lambda > 0$, there exists $R > 0$ such that: if $(t, U) \in \mathcal{C}^+$ and $t \leq \Lambda$, then $\|U\|_H \leq R$.*

Indeed, if the claim does not hold, there are $(U_n) \subset H$ and $(t_n) \subset [0, \Lambda]$ such that,

$$\|U_n\|_H \rightarrow \infty \quad \text{and} \quad U_n = F(t_n, U_n).$$

Consider $W_n = U_n/\|U_n\|_H$, where $W_n = (\bar{u}_n, \bar{v}_n)$ for $\bar{u}_n = u_n/\|U_n\|_H$ and $\bar{v}_n = v_n/\|U_n\|_H$. Thus,

$$\int_{\Omega} \nabla \bar{u}_n \nabla \varphi dx + \int_{\Omega} \phi_{(u_n, v_n)}(x) \bar{u}_n \varphi dx = t_n \int_{\Omega} (a \bar{u}_n + b \bar{v}_n) \varphi dx$$

and

$$\int_{\Omega} \nabla \bar{v}_n \nabla \eta dx + \int_{\Omega} \psi_{(u_n, v_n)}(x) \bar{v}_n \eta dx = t_n \int_{\Omega} (c \bar{u}_n + d \bar{v}_n) \eta dx$$

for all $\varphi, \eta \in H_0^1(\Omega)$. Once (W_n) is bounded in H , without loss of generality, we can suppose that there is $W \in H$, $W = (u, v)$, such that

$$\bar{u}_n \rightharpoonup u \text{ in } H_0^1(\Omega), \bar{u}_n \rightarrow u \text{ in } L^2(\Omega) \text{ and } \bar{u}_n(x) \rightarrow u(x) \text{ a.e. in } \Omega \quad (6)$$

$$\bar{v}_n \rightharpoonup v \text{ in } H_0^1(\Omega), \bar{v}_n \rightarrow v \text{ in } L^2(\Omega) \text{ and } \bar{v}_n(x) \rightarrow v(x) \text{ a.e. in } \Omega. \quad (7)$$

For $\varphi = \frac{\bar{u}_n}{\|U_n\|_H^\gamma}$ and $\eta = \frac{\bar{v}_n}{\|U_n\|_H^\gamma}$ as functions of test, and recalling that $t^\gamma \phi_{(u_n, v_n)} = \phi_{(tu_n, tv_n)}$ and also $t^\gamma \psi_{(u_n, v_n)} = \psi_{(tu_n, tv_n)}$, for all $t > 0$, getting

$$\frac{1}{\|U_n\|_H^\gamma} \|\bar{u}_n\|_{H_0^1(\Omega)}^2 + \int_{\Omega} \phi_{(\bar{u}_n, \bar{v}_n)} \bar{u}_n^2 dx = t_n \int_{\Omega} (a \bar{u}_n + b \bar{v}_n) \frac{\bar{u}_n}{\|U_n\|_H^\gamma} dx \quad (8)$$

$$\frac{1}{\|U_n\|_H^\gamma} \|\bar{v}_n\|_{H_0^1(\Omega)}^2 + \int_{\Omega} \psi_{(\bar{u}_n, \bar{v}_n)} \bar{v}_n^2 dx = t_n \int_{\Omega} (c \bar{u}_n + d \bar{v}_n) \frac{\bar{v}_n}{\|U_n\|_H^\gamma} dx. \quad (9)$$

Therefore, using Hölder inequality and (6) – (7) in (8) – (9),

$$\lim_{n \rightarrow \infty} \int_{\Omega} \phi_{(\bar{u}_n, \bar{v}_n)} \bar{u}_n^2 dx = \lim_{n \rightarrow \infty} \int_{\Omega} \psi_{(\bar{u}_n, \bar{v}_n)} \bar{v}_n^2 dx = 0. \quad (10)$$

From Fatou lemma,

$$\int_{\Omega} \phi_{(u, v)} u^2 dx \leq \lim_{n \rightarrow \infty} \int_{\Omega} \phi_{(\bar{u}_n, \bar{v}_n)} \bar{u}_n^2 dx = 0 \quad (11)$$

$$\int_{\Omega} \psi_{(u, v)} v^2 dx \leq \lim_{n \rightarrow \infty} \int_{\Omega} \psi_{(\bar{u}_n, \bar{v}_n)} \bar{v}_n^2 dx = 0. \quad (12)$$

And so,

$$\int_{\Omega \times \Omega} K(x, y) f(|u(y)|, |v(y)|) |u(x)|^2 dx dy = \int_{\Omega \times \Omega} \Gamma(x, y) g(|u(y)|, |v(y)|) |v(x)|^2 dx dy = 0. \quad (13)$$

Thus, by (f_1) ,

$$0 \leq \epsilon \int_{\Omega \times \Omega} K(x, y) |u(y)|^\gamma |u(x)|^2 dx dy \leq \int_{\Omega \times \Omega} K(x, y) f(|u(y)|, |v(y)|) |u(x)|^2 dx dy = 0 \quad (14)$$

$$0 \leq \epsilon \int_{\Omega \times \Omega} \Gamma(x, y) |v(y)|^\gamma |v(x)|^2 dx dy \leq \int_{\Omega \times \Omega} \Gamma(x, y) g(|u(y)|, |v(y)|) |v(x)|^2 dx dy = 0. \quad (15)$$

and, consequently

$$\int_{\Omega \times \Omega} K(x, y) |u(y)|^\gamma |u(x)|^2 dx dy = \int_{\Omega \times \Omega} \Gamma(x, y) |v(y)|^\gamma |v(x)|^2 dx dy = 0. \quad (16)$$

Since that $K, \Gamma \in \mathcal{K}$, we have that $u = v = 0$. Hence, (\bar{u}_n) and (\bar{v}_n) converge to 0 in $L^2(\Omega)$. Considering $\varphi = \bar{u}_n$ and $\eta = \bar{v}_n$ as test function, we get that

$$\int_{\Omega} |\nabla \bar{u}_n|^2 dx + \int_{\Omega} \phi_{(u_n, v_n)} \bar{u}_n^2 dx = t_n \int_{\Omega} (a \bar{u}_n + b \bar{v}_n) \bar{u}_n dx \quad (17)$$

$$\int_{\Omega} |\nabla \bar{v}_n|^2 dx + \int_{\Omega} \psi_{(u_n, v_n)} \bar{v}_n^2 dx = t_n \int_{\Omega} (c \bar{u}_n + d \bar{v}_n) \bar{v}_n dx. \quad (18)$$

As (t_n) is bounded by Λ ,

$$\int_{\Omega} |\nabla \bar{u}_n|^2 dx \leq \Lambda \left[a \int_{\Omega} |\bar{u}_n|^2 dx + b \int_{\Omega} |\bar{u}_n \bar{v}_n| dx \right] \quad (19)$$

$$\int_{\Omega} |\nabla \bar{v}_n|^2 dx \leq \Lambda \left[c \int_{\Omega} |\bar{u}_n \bar{v}_n| dx + d \int_{\Omega} |\bar{v}_n|^2 dx \right]. \quad (20)$$

Consequently, $\|W_n\|_H \rightarrow 0$. This contradicts the fact that $\|W_n\|_H = 1$ for all $n \in \mathbb{N}$, and the lemma follows.

Since (U_n) is bounded in H , iteration arguments imply that (U_n) is bounded in $L^\infty(\Omega) \times L^\infty(\Omega)$, and the proof is done. \blacksquare

Conclusion of the proof of Lemma 5.1 and proof of the Theorem 1.2

From Lemma 5.4, for all $t > t_1$, we have that $(\{t\} \times E) \cap \mathcal{C}^+ \neq \emptyset$, that is, \mathcal{C}^+ crosses the hyperplane $\{t\} \times E$. Indeed, otherwise there $\Lambda > t_1$ such that \mathcal{C}^+ does not cross the hyperplane $\{\Lambda\} \times E$, thus by Lemma 5.4 there exists $R > 0$ such that $(t, U) \in \mathcal{C}^+$, $t \in [0, \Lambda]$, and $\|U\| \leq R$. Therefore, \mathcal{C}^+ would be bounded, which contradicts the Lemma 5.3.

To finalize the proof of Lemma 5.1, we must show that there is no solution for (P_7) when $t \leq t_1 = \frac{\Delta}{\lambda}$. Indeed, arguing by contradiction, if (t, U) is a solution of (P_7) , with $t \leq t_1$ and $U = (u, v) > 0$, we have

$$\begin{cases} -\Delta u + \phi_{(u, v)} u = t[au + \frac{b}{\sigma}(\sigma v)], & \text{in } \Omega \\ -\Delta(\sigma v) + \psi_{(u, v)}(\sigma v) = t[(c\sigma)u + d(\sigma v)], & \text{in } \Omega \\ u = v = 0, & \text{on } \partial\Omega \end{cases}$$

for all $\sigma > 0$. In particular, if $\sigma^2 = \frac{b}{c}$, may we fix $w = \sigma v$ and observe that $\hat{b} := \frac{b}{\sigma} = c\sigma$. And so, $\hat{U} = (u, w) \in E$ is solution of the problem

$$\begin{cases} -\Delta u + \phi_{(u,v)} u = t[au + \hat{b}w], & \text{in } \Omega \\ -\Delta w + \psi_{(u,v)} w = t[\hat{b}u + dw], & \text{in } \Omega \\ u, w > 0, & \text{in } \Omega \\ u = w = 0, & \text{on } \partial\Omega \end{cases}$$

Since $A_0 = \begin{pmatrix} a & \hat{b} \\ \hat{b} & d \end{pmatrix}$ is a symmetric matrix, we know that

$$\mu|z|^2 \leq \langle A_0 z, z \rangle \leq \lambda|z|^2, \quad \text{for all } z \in \mathbb{R}^2. \quad (21)$$

On the other hand,

$$\int_{\Omega} \left\langle tA_0 \begin{pmatrix} u \\ w \end{pmatrix}, \begin{pmatrix} u \\ w \end{pmatrix} \right\rangle dx = \int_{\Omega} |\nabla u|^2 + \phi_{(u,v)}(x)u^2 dx + \int_{\Omega} |\nabla w|^2 + \psi_{(u,v)}(x)w^2 dx > \int_{\Omega} (|\nabla u|^2 + |\nabla w|^2) dx \quad (22)$$

and, consequently by (21),

$$\int_{\Omega} (|\nabla u|^2 + |\nabla w|^2) dx < t\lambda \int_{\Omega} (|u|^2 + |w|^2) dx. \quad (23)$$

On the other hand, by Poincaré's inequality

$$\lambda_1 \int_{\Omega} (|u|^2 + |w|^2) dx < t\lambda \int_{\Omega} (|u|^2 + |w|^2) dx. \quad (24)$$

Hence, $t > \frac{\lambda_1}{\lambda}$ which is a contradiction. This proves the lemma.

In relation to Theorem 1.2, by Lemma 5.1, it is clear that (P_1) has solution if, and only if, $t_1 = \frac{\lambda_1}{\lambda} < 1$. Therefore, (P_1) has solution if, and only if, $\lambda > \lambda_1$. This proves the theorem.

6 Proof of Theorem 1.3

As was done in the Theorem 1.2, to prove the Theorem 1.3 via bifurcation theory, it is necessary to introduce a parameter $t > 0$ in the problem (P_2) and prove the lemma below:

Lemma 6.1. *Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ a matrix such that: there is a positive and largest eigenvalue of A that is the unique positive eigenvalue λ with an eigenvector $z > 0$ and $\dim N(\lambda I - A) = 1$. Then, the problem*

$$\begin{cases} -\Delta U + \phi(x)U = tAU, & \text{in } \Omega \\ U > 0, & \text{in } \Omega \\ U = 0, & \text{on } \partial\Omega \end{cases} \quad (P_8)$$

has solution for all $t > t_1$, where $t_1 = \frac{\lambda_1}{\lambda}$ and λ_1 is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$.

To prove the lemma above, it is necessary to note that: using the definitions of S and G , it is easy to check that $(t, U) \in \mathbb{R} \times E$ solves (P_8) if, and only if,

$$U = F(t, U) = tS(U) + G(U).$$

In the sequel, we will apply again the result due to Rabinowitz [14], for prove the Lemma 6.1.

Again, by study done in Subsection 3.1, a eigenfunction U_1 associated with eigenvalue $t_1 = \lambda_1/\lambda$ of the linear problem can be chosen positive. In addition, t_1^{-1} is an eigenvalue of multiplicity 1 for S . From global bifurcation theorem (Theorem 5.1), there exists a closed connected component $\mathcal{C} = \mathcal{C}_{t_1}$ of solutions for (P_8) , which satisfies (i) or (ii), of the Theorem 5.1. We claim that (ii) does not occur. In order to show this claim, we need the lemma below

Lemma 6.2. *There exists $\delta > 0$ such that, if $(t, U) \in \mathcal{C}$ with $|t - t_1| + \|U\| < \delta$ and $U \neq 0$, then U has defined signal, that is,*

$$U(x) > 0, \quad \forall x \in \Omega \quad \text{or} \quad U(x) < 0, \quad \forall x \in \Omega.$$

that is, the same Lemma 5.2, but now on the problem (P_8) . The demonstration is absolutely analogous with trivial modifications.

A trivial fact in the previous case is the decomposition of \mathcal{C} in $\mathcal{C}^+ \cup \mathcal{C}^-$. Here, to see such decomposition, a special attention is required.

In order to achieve this decomposition, it is necessary to introduce an auxiliary operator, whose properties are similar to the Laplacian operator.

Auxiliary operator

By fixing $\psi \in L^\infty(\overline{\Omega})$, the solution operator $S_L : L^2(\Omega) \rightarrow L^2(\Omega)$ such that $S_L(v) = u$, where u is the unique weak solution for the linear problem

$$\begin{cases} L(u) = v, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \quad (25)$$

where $L(u) = -\Delta u + \psi(x)u$. This solution operator is compact self-adjoint, then by spectral theory there exists a complete orthonormal basis $\{\phi_n\}$ of $L^2(\Omega)$ and a corresponding sequence of positive real numbers $\{\lambda_n\}$ with $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$ such that

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots$$

and

$$\begin{cases} L(\phi_n) = \lambda_n \phi_n, & \text{in } \Omega \\ \phi_n = 0, & \text{on } \partial\Omega. \end{cases}$$

Moreover, using Lagrange multiplier it is possible to prove the following characterization for λ_1

$$\lambda_1 = \inf_{v \in H^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} [|\nabla v|^2 + \psi(x)v^2] dx}{\int_{\Omega} v^2 dx}. \quad (26)$$

The above identity is crucial to show that λ_1 is a simple eigenvalue and that a corresponding eigenfunction ϕ_1 can be chosen positive in Ω . Note that the Lemma 3.1 is valid, replacing $-\Delta U$ by LU and $\sigma(-\Delta)$ by $\sigma(L)$, where $LU = (L(u), L(v))$.

With the notations and properties introduced above, we have the following version of the Hopf's Lemma in matricial format:

Lemma 6.3. *If the problem*

$$\begin{cases} LU = AU, & \text{in } \Omega \\ U = 0, & \text{on } \partial\Omega \end{cases}$$

has solution U with $U \geq 0$ and $U \neq 0$, then $\sigma(L) \cap \sigma(A) \neq \emptyset$. Moreover, $U > 0$ in Ω and $\frac{\partial u}{\partial \eta}, \frac{\partial v}{\partial \eta} < 0$ on $\partial\Omega$.

Proof. Using the same arguments of the Lemma 3.1, we conclude that $\sigma(L) \cap \sigma(A) \neq \emptyset$. Furthermore, as there is a positive and largest eigenvalue of A that is the unique positive eigenvalue λ with an eigenvector $z > 0$ and $\dim N(\lambda I - A) = 1$, we have, by analogous argument to the Corollary 3.1, $\lambda = \lambda_1$ and $U = \phi_1 w$, where w is multiple of z . Moreover, we have that $U > 0$ and $\frac{\partial u}{\partial \eta}, \frac{\partial v}{\partial \eta} < 0$ on $\partial\Omega$. This completes the proof. ■

We can now prove the theorem:

Lemma 6.4. *Consider the sets*

$$\mathcal{C}^+ = \{(t, U) \in \mathcal{C} : U(x) > 0, \quad \forall x \in \Omega\} \cup \{(t_1, 0)\}$$

and

$$\mathcal{C}^- = \{(t, U) \in \mathcal{C} : U(x) < 0, \quad \forall x \in \Omega\} \cup \{(t_1, 0)\}.$$

Then,

$$\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-. \quad (27)$$

Moreover, note that $\mathcal{C}^- = \{(t, U) \in \mathcal{C} : (t, -U) \in \mathcal{C}^+\}$, $\mathcal{C}^+ \cap \mathcal{C}^- = \{(t_1, 0)\}$ and \mathcal{C}^+ is unbounded if, and only if, \mathcal{C}^- is also unbounded.

Proof. Of course, the proof is complete, showing that \mathcal{C}^+ is closed and open. For $(t, U) \in \overline{\mathcal{C}^+}$, we have $U \neq 0$ and $U \geq 0$, where $U = (u, v)$. As,

$$\begin{cases} -\Delta u + \phi_{(u,v)}u = au + bv, & \text{in } \Omega \\ -\Delta v + \phi_{(u,v)}v = cu + dv, & \text{in } \Omega \\ u = v = 0, & \text{on } \partial\Omega \end{cases} \quad (28)$$

we get,

$$\begin{cases} LU = tAU, & \text{in } \Omega \\ U = 0, & \text{on } \partial\Omega \end{cases} \quad (29)$$

for $L(w) := -\Delta w + \phi(x)w$, where $\phi(x) = \phi_{(u,v)}(x)$. Therefore, by Lemma 6.3, we obtain $U > 0$ in Ω and, consequently, \mathcal{C}^+ is closed. Now, for $(t, U) \in \mathcal{C}^+$, we have, by Lemma 6.3, that $U > 0$ and $\frac{\partial u}{\partial \eta}, \frac{\partial v}{\partial \eta} < 0$ on $\partial\Omega$. Therefore, by Hopf's Lemma, $(t, U) \in \text{int}\mathcal{C}^+$. This proves the lemma. ■

Following the same steps of the previous section, we should prove:

Lemma 6.5. \mathcal{C}^+ is unbounded.

But, the proof is absolutely analogous, with some trivial modifications, to the proof of the Lemma 5.3, so not we shall prove. The same can be said of the a priori estimate:

Lemma 6.6. (*A priori estimate*) For any $\Lambda > 0$, there exists $R > 0$ such that, if $(t, U) \in \mathcal{C}^+$ and $t \in [0, \Lambda]$, then $\|U\| \leq R$.

Conclusion of the proof of Lemma 6.1 and proof of the Theorem 1.3

From Lemma 6.6, for all $t > t_1$, we have that $(\{t\} \times E) \cap \mathcal{C}^+ \neq \emptyset$, that is, \mathcal{C}^+ crosses the hyperplane $\{t\} \times E$. Indeed, otherwise there $\Lambda > t_1$ such that \mathcal{C}^+ does not cross the hyperplane $\{\Lambda\} \times E$, thus by Lemma 6.6 there exists $R > 0$ such that $(t, U) \in \mathcal{C}^+$, $t \in [0, \Lambda]$ and $\|U\| \leq R$. Therefore, \mathcal{C}^+ would be bounded, which contradicts the Lemma 6.5.

In relation to Theorem 1.3, by Lemma 6.1, it is clear that (P_2) has solution if $t_1 = \frac{\lambda_1}{\lambda} < 1$. Therefore, (P_2) has solution if $\lambda > \lambda_1$. This proves the theorem.

7 Appendix

This section is dedicated to present some details that could be removed from the text without prejudice to the understanding of the content.

7.1 Proof of the Lemmas 3.1 and 3.2

Proof. (Lemma 3.1)

Suppose that, $\int_{\Omega} u\phi_j dx \neq 0$. Multiplying the equations in (Q_1) by ϕ_j and integrating on Ω , we get:

$$\lambda_j \int_{\Omega} u\phi_j dx = \int_{\Omega} \nabla u \nabla \phi_j dx = a \int_{\Omega} u\phi_j dx + b \int_{\Omega} v\phi_j dx$$

and

$$\lambda_j \int_{\Omega} v\phi_j dx = \int_{\Omega} \nabla v \nabla \phi_j dx = c \int_{\Omega} u\phi_j dx + d \int_{\Omega} v\phi_j dx,$$

in the matricial form,

$$Az = \lambda_j z, \text{ where } z = \begin{pmatrix} \int_{\Omega} u\phi_j dx \\ \int_{\Omega} v\phi_j dx \end{pmatrix}.$$

That is, λ_j is eigenvalue of A with eigenvector (nonzero) $z = \begin{pmatrix} \int_{\Omega} u\phi_j dx \\ \int_{\Omega} v\phi_j dx \end{pmatrix}$.

Therefore, A has at most two eigenvalues of $(-\Delta, H_0^1(\Omega))$. If A has two eigenvalues of $(-\Delta, H_0^1(\Omega))$, λ_j and λ_m , we have

$$\int_{\Omega} u\phi_k dx = 0, \text{ for all } k \neq j, m.$$

We conclude that there are, in the maximum, two eigenvalues, consider λ_j and λ_m such that $u = \alpha_1\phi_j + \beta_1\phi_m$ and $v = \alpha_2\phi_j + \beta_2\phi_m$, where

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \text{ is eigenvector of } A \text{ associated with the eigenvalue } \lambda_j$$

and

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \text{ is eigenvector of } A \text{ associated with the eigenvalue } \lambda_m.$$

The itens (ii) and (iii), follow analyzing what has been done up.

All details can be found in [10], doctoral thesis that deals with this subject in detail. ■

Now, suppose that (Q_1) admits a nonnegative and nonzero solution, in the sense $U \geq 0$ if $u \geq 0$ and $v \geq 0$. By demonstration of the above lemma, we have $u = \alpha_1\phi_j + \beta_1\phi_m$ and $v = \alpha_2\phi_j + \beta_2\phi_m$. We claim

that, $j = 1$ or $m = 1$. Indeed, otherwise, as $\int_{\Omega} u\phi_1 dx > 0$ and from orthogonality of the eigenfunctions associated with distinct eigenvalues of $(-\Delta, H_0^1(\Omega))$, follow that

$$0 < \int_{\Omega} u\phi_1 dx = \alpha_1 \int_{\Omega} \phi_j \phi_1 dx + \beta_1 \int_{\Omega} \phi_m \phi_1 dx = 0$$

which is a absurd.

Suppose that $j = 1$. Thus, $u = \alpha_1 \phi_1 + \beta_1 \phi_m$ and $v = \alpha_2 \phi_1 + \beta_2 \phi_m$, so

$$0 < \int_{\Omega} u\phi_1 dx = \alpha_1 \int_{\Omega} \phi_1^2 dx$$

that is, $\alpha_1 > 0$. Analogously, $\alpha_2 > 0$. And, consequently, there is an eigenvector of A , associated with λ_1 , having both positive coordinates.

From these comments, we have the proof of the Lemma 3.2. Futhermore, recalling that $\frac{\partial \phi_1}{\partial \eta} < 0$ on $\partial\Omega$, we have the proof of the Corollary 3.1.

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